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TOPICAL ISSUE

An Optimal Choice of Characteristic Polynomial Roots for Pole Placement Control Design

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Abstract—The problem of finding the arrangement of closed-loop control system poles that minimizes an objective function is considered. The system optimality criterion is the value of the H_∞ norm of the frequency transfer function relative to the disturbance with constraints imposed on the system pole placement and the values of the H_∞ norm of the sensitivity function and the transfer function from measurement noise to control. An optimization problem is formulated as follows: the vector of variables consists of the characteristic polynomial roots of the closed loop system with the admissible values restricted to a given pole placement region; in addition to the optimality criterion, the objective function includes penalty elements for other constraints. It is proposed to use a logarithmic scale for the moduli of the characteristic polynomial roots as elements of the vector of variables. The multi-extremality problem of the objective function is solved using the multiple start procedure. A coordinate descent modification with a pair of coordinates varied simultaneously is used for search.

Keywords: control design, transfer function, pole placement, optimization, robust system

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1. INTRODUCTION

Rejection of an unmeasurable disturbance is one of the main tasks of control design [1]. On the other hand, the resulting system must satisfy robustness conditions since the plant model used for control design is inaccurate. For linear systems, first of all, the requirements for stability margins must be met [2]. These requirements can be specified as the minimum acceptable stability margins radius [3] or limiting the value of the sensitivity function [4, 5]. The H_∞ norm of the measurement noise sensitivity function can serve as a measure of robustness to unmodeled dynamics [5, 6].

Many control design techniques lead to an optimization problem. For example, the methods of H_∞ optimization [7] and invariant ellipsoids [1] reduce to an optimization procedure for solving a system of linear matrix inequalities. If the variables are the coefficients of a fixed-structure controller, the optimization problem may become non-convex and multi-extremal [8, 9]. The successful results of solving such problems allowed developing similar approaches for tuning PID controllers widely used in practice [10, 11].

For a linear single-input single-output (SISO) system, the following idea of optimization of the closed-loop system pole placement was proposed in [12]: the controller coefficients are found via the standard pole placement procedure, and the roots of the desired characteristic polynomial of the closed loop system are searched using an optimization procedure for specified quality criteria and constraints. The standard global optimization procedure from the MATLAB Global Optimization Toolbox [13] was used in [12]. The value of the H_∞ norm of the transfer function relative to the disturbance was chosen as a quality criterion under given constraints on the values of the H_∞ norms of the sensitivity function and the transfer function from measurement noise to control. In addition, constraints were imposed on the system pole placement.

This article is devoted to developing an optimization procedure for finding an optimal closed-loop system pole placement that minimizes a given objective function subject to specified constraints; in the corresponding optimization problem, the vector of variables consists of the characteristic polynomial roots of the closed loop system.

2. PROBLEM STATEMENT

Consider a linear SISO system whose structure is presented in Fig. 1. Let the plant be described by the transfer function

$$P(s) = \frac{b(s)}{a(s)} = \frac{b_{n-1}s^{n-1} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}, \tag{1}$$

where s is the Laplace transform variable; the coefficients $a_i, b_i \in \mathbb{R}$ $(i = 0, \ldots, n-1)$ have known values, and at least one of the coefficients b_i is nonzero; the polynomials $a(s)$ and $b(s)$ are coprime. The frequency response function is obtained for $s = j\omega$, where $\omega \in [0, \infty)$. By assumption, as frequency response functions are used, all system signals (including the unmeasured exogenous disturbance) are integrable and satisfy the restrictions for applying the Fourier transform [2]:

$$\int_{-\infty}^{+\infty} |f(t)|dt < \infty.$$

Suppose that the controller's transfer function has the form

$$C(s) = \frac{d(s)}{c(s)} = \frac{d_{n-1}s^{n-1} + \cdots + d_0}{c_{n-1}s^{n-1} + \cdots + c_0}, \tag{2}$$

where the controller order $(n-1)$ is determined by the order of the plant model (1). A higher-order controller, which can be constructed, e.g., by adding an integral component to the controller, is not considered here. A lower-order controller cannot be constructed by the pole placement technique; see the explanation below.

According to the pole placement method [14, 15], the polynomials $c(s)$ and $d(s)$ of the controller (2) can be obtained by solving the equation

$$a(s)c(s) + b(s)d(s) = \delta(s), \tag{3}$$

where the left-hand side is the characteristic polynomial of system (1), (2) in which $a(s)$ and $b(s)$ are the known polynomials of the plant's transfer function, and $\delta(s)$ is a given desired characteristic polynomial. As is known [14], there exists a unique solution of this equation under the condition $\deg d(s) < \deg a(s)$ or $\deg c(s) < \deg b(s)$. In addition, under the condition $\deg \delta(s) \geqslant 2 \deg a(s) - 1$,

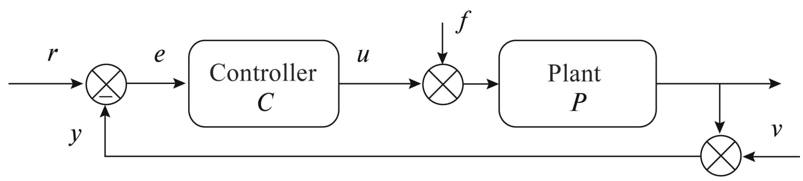

Fig. 1. Closed loop system: y—measured output, ν—measurement noise, r—reference signal, e—control error, u—control, and f—disturbance.

the causality of control is satisfied: $\deg d(s) \leqslant \deg c(s)$. Then, by choosing the desired polynomial $\delta(s)$ of degree $\deg \delta(s) = 2\deg a(s) - 1$, we obtain the solution (2) for which the conditions $\deg d(s) \leqslant \deg c(s)$ and $\deg d(s) < \deg a(s)$ hold. In this case, equation (3) can be solved by compiling a system of $2n$ linear algebraic equations with $2n$ unknowns when equating the coefficients of the left- and right-hand sides of equation (3) at the equal powers of s:

$$\begin{bmatrix} c_{n-1} \\ \dots \\ c_0 \\ d_{n-1} \\ \dots \\ d_0 \end{bmatrix} = W^{-1} \begin{bmatrix} \delta_{2n-1} \\ \dots \\ \delta_0 \end{bmatrix}, \tag{4}$$

where $W \in \mathbb{R}^{2n\times 2n}$ is a matrix obtained from the coefficients a_i, b_i $(i = 0, \dots, n-1)$.

Thus, for any plant (1), one can find a controller of the form (2) ensuring any given characteristic polynomial $\delta(s)$ of degree $(2n-1)$ for the closed loop system. Note that for an unstable plant of order n, there may not exist a controller of order below $(n-1)$ ensuring at least the stability of the system. Therefore, we consider a controller of order $(n-1)$ to ensure not only stability but also other system properties of the system by choosing an appropriate desired characteristic polynomial.

The characteristic polynomial can be represented as

$$\delta(s) = \prod_{i=1}^{n_r} (s + \lambda_i) \prod_{k=1}^{n_c} (s^2 + 2\zeta_k \breve{\omega}_k s + \breve{\omega}_k^2), \tag{5}$$

where $n_r = 2n - 2n_c - 1$ is the number of real roots of the polynomial $\delta(s)$ and n_c is the number of complex conjugate pairs of the roots; the values $\lambda_i, \breve{\omega}_k \in \mathbb{R}$ and $\zeta_k \in [0, 1]$ determine the closed-loop pole placement and the coefficients $\delta_0, \dots, \delta_{2n-2}$ in (4) while $\delta_{2n-1} = 1$. Let $\breve{\omega}_k$ denote the natural frequencies of the system since the notation ω is used for the frequency variable in transfer functions.

In addition to the standard constraints $\lambda_i > 0$, $\breve{\omega}_k > 0$, and $0 < \zeta_k \leqslant 1$, which ensure the stability of the closed loop system, it is possible to specify the supplementary ones

$$0 < \lambda_{\min} \leqslant \lambda_i \leqslant \lambda_{\max}, \quad 0 < \breve{\omega}_{\min} \leqslant \breve{\omega}_k \leqslant \breve{\omega}_{\max}, \quad 0 < \zeta_{\min} \leqslant \zeta_k \leqslant 1 \tag{6}$$

to obtain the desired speed and damping rate of the system and limit the high-frequency components.

Similar to [12], the value of H_∞ norm of the frequency response function relative to the disturbance is taken as the system quality criterion:

$$\|G_{yf}(j\omega)\|_\infty = \sup_\omega \left| \frac{b(j\omega)c(j\omega)}{\delta(j\omega)} \right|. \tag{7}$$

Moreover, the following constraints must be satisfied:

— for the H_∞ norm of the sensitivity function, the inequality

$$\|S(j\omega)\|_\infty = \sup_\omega \left| \frac{a(j\omega)c(j\omega)}{\delta(j\omega)} \right| \leqslant S_{\max} \tag{8}$$

to ensure the required stability margins;

— for the H_∞ norm of the frequency response function relative to the noise, the inequality

$$\|G_{u\nu}(j\omega)\|_\infty = \sup_\omega \left|\frac{a(j\omega)d(j\omega)}{\delta(j\omega)}\right| \leqslant N_{\max} \tag{9}$$

to ensure the robustness of the system in the presence of unmodeled dynamics by limiting the controller gain [5, 6].

Thus, the problem is to find a controller of the form (2) that minimizes the exogenous disturbance effect on the the plant (1) in terms of the norm (7) subject to the constraints (6), (8), and (9) under given values a_i, b_i $(i = 0, \ldots, n-1), \lambda_{\min}, \lambda_{\max}, \breve{\omega}_{\min}, \breve{\omega}_{\max}, \zeta_{\min}, S_{\max}$, and $N_{\max}$. It can be formulated as an optimization problem.

Problem 1. Find

$$\min_{x\in Q} \|G_{yf}(j\omega, x)\|_\infty$$

subject to

$$\begin{aligned} &\|S(j\omega, x)\|_\infty \leqslant S_{\max},\\ &\|G_{u\nu}(j\omega, x)\|_\infty \leqslant N_{\max}, \end{aligned} \tag{10}$$

where $S_{\max}$ and $N_{\max}$ are given values. The vector of variables $x \in \mathbb{R}^{2n-1}$ has the form

$$x = [\lambda_1, \ldots, \lambda_{n_r}, \breve{\omega}_1, \ldots, \breve{\omega}_{n_c}, \zeta_1, \ldots, \zeta_{n_c}], \tag{11}$$

where n_r and n_c are given values such that $0 \leqslant n_c \leqslant n-1, n_r = 2n - 2n_c - 1$, and n is a known order of the plant (1). The admissible region Q is determined by inequalities (6) with the given parameters $\lambda_{\min}, \lambda_{\max}, \breve{\omega}_{\min}, \breve{\omega}_{\max}$, and $\zeta_{\min}$. In accordance with (7)–(9), the frequency response functions $G_{yf}(j\omega, x), S(j\omega, x)$, and $G_{u\nu}(j\omega, x)$ are constructed from the given polynomials $a(j\omega)$ and $b(j\omega)$ of the plant (1), the polynomial $\delta(j\omega)$ determined for the vector (11) by formula (5), and the controller polynomials $c(j\omega)$ and $d(j\omega)$ whose coefficients are found by solving system (4).

Note that the constraints (6), (8), and (9) may be not satisfied simultaneously; in this case, the set of admissible values will be empty. This issue is not considered here: the constraints are assumed to be consistent. For a particular problem, an iterative process can be carried out in practice to find acceptable values of the constraints for reaching an acceptable value of the objective function.

3. SEARCH FOR THE OPTIMAL ROOTS OF THE CHARACTERISTIC POLYNOMIAL

3.1. Objective Function with Penalties

We use the penalty function method to satisfy the constraints. For the value $\|G(j\omega, x)\|_\infty$, the penalty function $\tilde{G}(x)$ is given by

$$\tilde{G}(x) = \begin{cases} 0 \text{ if } \|G(j\omega, x)\|_\infty \leqslant G_{\max} \\ \ln \dfrac{\|G(j\omega, x)\|_\infty}{G_{\max}} \;\text{ if } \|G(j\omega, x)\|_\infty > G_{\max}. \end{cases} \tag{12}$$

In this case, the objective function takes the form

$$f(x) = \|G_{yf}(j\omega, x)\|_\infty + \mu_1\tilde{S}(x) + \mu_2\tilde{G}_{u\nu}(x), \tag{13}$$

where $\mu_1 > 0$ and $\mu_2 > 0$ are weight coefficients, and $\tilde{S}(x)$ and $\tilde{G}_{u\nu}(x)$ are the penalty functions obtained using (12) for the constraints (10). Note that due to (12), the objective function (13) is non-differentiable at the points where $\|S(j\omega, x)\|_\infty = S_{\max}$ or $\|G_{u\nu}(j\omega, x)\|_\infty = N_{\max}$. Moreover, the functions (7)–(9) may be non-convex and multi-extremal, and their gradients are not written in explicit form.

3.2. Scaling of the Variables

The logarithmic scale is often used to analyze dynamic systems in the frequency domain [2]. Note that the elements λ_i and $\breve{\omega}_k$ of the vector of variables (11) are the natural frequencies of the system. We convert them to a logarithmic scale, thus assigning a greater weight to changes in the roots with a modulus close to zero (slow system dynamics) compared to changes in those with a large modulus (fast system dynamics):

$$\begin{aligned} \tilde{x} &= [\lg\lambda_1, \ldots, \lg\lambda_{n_r}, \lg\breve{\omega}_1, \ldots, \lg\breve{\omega}_{n_c}, \zeta_1, \ldots, \zeta_{n_c}] \\ &= [\tilde{\lambda}_1, \ldots, \tilde{\lambda}_{n_r}, \tilde{\omega}_1, \ldots, \tilde{\omega}_{n_c}, \zeta_1, \ldots, \zeta_{n_c}], \end{aligned} \tag{14}$$

where $\tilde{\lambda}_i$ and $\tilde{\omega}_i$ are the common logarithms of the variables λ_i and $\breve{\omega}_i$, respectively. In this case, the constraints (6) take the form

$$\begin{aligned} 0 < \lg\lambda_{\min} \leqslant \tilde{\lambda}_i \leqslant \lg\lambda_{\max}, \\ 0 < \lg\breve{\omega}_{\min} \leqslant \tilde{\omega}_k \leqslant \lg\breve{\omega}_{\max}, \\ 0 < \zeta_{\min} \leqslant \zeta_k \leqslant 1. \end{aligned} \tag{15}$$

To calculate the objective function, the values of the variables must be rescaled to (11) by raising to the tenth power: $\lambda_i = 10^{\tilde{\lambda}_i}, i = 1, \ldots, n_r$, and $\breve{\omega}_i = 10^{\tilde{\omega}_i}, i = 1, \ldots, n_c$. The notations without the subscripts, $\tilde{\lambda}, \tilde{\omega}$, and ζ, will be used for the corresponding groups in the vector of variables (14):

$$\begin{aligned} \tilde{\lambda} &= [\tilde{\lambda}_1, \ldots, \tilde{\lambda}_{n_r}], \\ \tilde{\omega} &= [\tilde{\omega}_1, \ldots, \tilde{\omega}_{n_c}], \\ \zeta &= [\zeta_1, \ldots, \zeta_{n_c}]. \end{aligned}$$

Accordingly, the vector (14) will be represented as $\tilde{x} = [\tilde{\lambda}, \tilde{\omega}, \zeta]$.

The dynamics with frequencies exceeding manyfold the minimum natural frequency of the control plant are often neglected during system design. Therefore, the difference between the common logarithms of the admissible values of the moduli of the characteristic polynomial roots usually is not greater than 5. For example, when considering a system with slow dynamics and $\lambda_{\min} = 0.001$ and $\lambda_{\max} = 1$, we obtain $\lg\lambda_{\min} = -3$ and $\lg\lambda_{\max} = 0$; for a system with fast dynamics, $\lg\lambda_{\min} = 2$ and $\lg\lambda_{\max} = 6$ under the same or similar values for $\lg\breve{\omega}_{\min}$ and $\lg\breve{\omega}_{\max}$. Then the choice of the minimum step for the groups of variables $\tilde{\lambda}$ and $\tilde{\omega}$ is obvious. It follows from practical considerations that a step from 0.0001 to 0.01 will be quite small under such scales. This step is also reasonable for the group ζ, whose elements belong to the range $[\zeta_{\min}, 1]$.

3.3. Multiple Start

Multiple start is a standard approach to settling the multi-extremality problem of the objective function (13): the search procedure is executed from different initial points. For the problem under consideration, the initial values can be chosen, e.g., using the following rule:

— Choose the number of alternatives n_1, n_2, and n_3 for the groups of variables $\tilde{\lambda}, \tilde{\omega}$, and ζ, respectively.

— For the groups $\tilde{\lambda}$ and $\tilde{\omega}$, create alternatives in which the first elements of the groups are uniformly distributed in the admissible range and the remaining elements are uniformly distributed in the range $[\tilde{\lambda}_1, \lg\lambda_{\max}]$ or $[\tilde{\omega}_1, \lg\breve{\omega}_{\max}]$, respectively:

$$\begin{aligned}
\tilde{\lambda}_1^{(\ell)} &= \lg\lambda_{\min} + \ell\frac{\lg\lambda_{\max} - \lg\lambda_{\min}}{n_1+1},\ \ell = 1,\dots,n_1,\\
\tilde{\lambda}_i^{(\ell)} &= \tilde{\lambda}_1^{(\ell)} + (i-1)\frac{\lg\lambda_{\max} - \tilde{\lambda}_1^{(\ell)}}{n_r},\ i = 2,\dots,n_r,\\
\tilde{\omega}_1^{(\ell)} &= \lg\breve{\omega}_{\min} + \ell\frac{\lg\breve{\omega}_{\max} - \lg\breve{\omega}_{\min}}{n_2+1},\ \ell = 1,\dots,n_2,\\
\tilde{\omega}_i^{(\ell)} &= \tilde{\omega}_1^{(\ell)} + (i-1)\frac{\lg\breve{\omega}_{\max} - \tilde{\omega}_1^{(\ell)}}{n_c},\ i = 2,\dots,n_c.
\end{aligned} \tag{16}$$

— Use the same values for all elements of the group ζ :

$$\zeta_i^{(\ell)} = \begin{cases} \dfrac{1-\zeta_{\min}}{2} & \text{if } n_3 = 1 \\ \zeta_{\min} + (\ell-1)\dfrac{1-\zeta_{\min}}{n_3-1} & \text{if } n_3 > 1, \end{cases} \qquad i = 1,\dots,n_c,\ \ell = 1,\dots,n_3. \tag{17}$$

— Create the set of $n_1 \cdot n_2 \cdot n_3$ initial points by combining all alternatives for each group.

For example, 32 initial points will be obtained if $n_1 = 4, n_2 = 4$, and $n_3 = 2$.

When building another grid of the initial values, one should keep in mind the following: the rearrangement of any elements within the groups $\tilde{\lambda}$ and $\tilde{\omega}$ makes no sense because, due to (5), the resulting polynomial $\delta(s)$ will be the same regardless of the order of the elements in the group.

3.4. Search Method

The objective function (13) is generally non-convex, multi-extremal, and non-differentiable at some points; therefore, standard search methods will not necessarily find a global minimum. For the problem under consideration, we use a combined method in which coordinate descent is applied for the group of variables ζ whereas the groups $\tilde{\lambda}$ and $\tilde{\omega}$ are merged to execute the search procedure by the pairs of coordinates. The dimension of the vector $[\tilde{\lambda}, \tilde{\omega}]$ equals $n_a = n_r + n_c$, and $n_a!/(2(n_a-2)!)$ pairs can be made from the elements of this vector. For $n_a = 10$, we have 45 pairs, which is computationally feasible. For most practical 1D problems, this restriction will be satisfied; for higher-dimension problems, however, some pairs should be discarded. For example, only neighbor elements can be combined into pairs, which gives $(n_a - 1)$ pairs; alternatively, pairs can be formed separately for the groups $\tilde{\lambda}$ and $\tilde{\omega}$.

We determine the next point $(k+1)$ after varying a pair of elements i, j $(i = 1,\dots,n_a-1,\ j = i+1,\dots,n_a)$ as follows:

$$\tilde{x}_{k+1} = \arg\min_{\alpha,\beta} f(\tilde{x}_k + \alpha e_i + \beta e_j), \tag{18}$$

where e_i and e_j are the vectors with ones for elements i and j, respectively, and zeros for all other elements; α and β are values from some set of variations, e.g.,

$$\alpha, \beta \in \{0,\ 0.001,\ -0.001,\ 0.01,\ -0.01\}. \tag{19}$$

If the result of (18) is $\alpha = \beta = 0$, then a new point has not been obtained. If a new record value of the objective function is reached, then the 1D search procedure can be executed for the corresponding values α and β :

$$\tilde{x}_{k+1} = \arg\min_{\gamma} f(\tilde{x}_k + \gamma\alpha e_i + \gamma\beta e_j), \tag{20}$$

where, e.g., $\gamma \in \{0, 10\}$.

Fixed steps are used due to the nonconvexity of the objective function: finding an optimal step in a given direction may be a computationally difficult task.

When varying the elements of the groups $\tilde{\lambda}$ and $\tilde{\omega}$, we take into account that the objective function is independent of the rearrangement of these elements. Therefore, it is possible to fix an order of elements $\tilde{\lambda}_1 \leqslant \tilde{\lambda}_2 \leqslant \ldots \leqslant \tilde{\lambda}_{n_r}$, $\tilde{\omega}_1 \leqslant \tilde{\omega}_2 \leqslant \ldots \leqslant \tilde{\omega}_{n_c}$ and, in addition to the bounds (15), use neighbor elements as bounds as well. For example, in the case $n_r > 2$,

$$\begin{aligned} &\tilde{\lambda}_1 \in [\lg \lambda_{\min}, \tilde{\lambda}_2], \\ &\tilde{\lambda}_i \in [\tilde{\lambda}_{i-1}, \tilde{\lambda}_{i+1}],\ 1 < i < n_r, \\ &\tilde{\lambda}_{n_r} \in [\tilde{\lambda}_{n_r-1}, \lg \lambda_{\max}]. \end{aligned} \tag{21}$$

After the search procedure (18) for all pairs $(i = 1, \ldots, n_a - 1,\ j = i + 1, \ldots, n_a)$, we execute coordinate descent for the group ζ :

$$\tilde{x}_{k+1} = \arg\min_{\eta} f(\tilde{x}_k + \eta e_i),\ i = 1, \ldots, n_c, \tag{22}$$

where η is the set of fixed steps and e_i is the vector with one for element $(i + n_r + n_c)$ and zeros for the other elements. For example, the set of steps can be

$$\eta \in \{0.001, -0.001, 0.01, -0.01, 0.05, -0.05\}. \tag{23}$$

The elements of the group ζ are varied within the specified bounds: $\zeta_i \in [\zeta_{\min}, 1]$.

Thus, Problem 1 is solved using the following algorithm for $n_a > 1$.

Algorithm 1.

1. Choose the penalty weight coefficients μ_1 and μ_2 for the objective function (13) and set the search threshold ε.

2. Generate a grid of initial points as described in subsection 3.3 and take the first initial point.

3. Calculate the value of the objective function at the initial point, $f^{(\ell)}_{\min}$.

4. Take a pair of elements from the groups of variables $\tilde{\lambda}$ and $\tilde{\omega}$.

5. Execute (18) through the exhaustive search procedure over the set (19).

6. If a new record value of the objective function is obtained, execute (20) in the obtained direction and go to the new point.

7. Take the next pair of elements from the groups of variables $\tilde{\lambda}$ and $\tilde{\omega}$ and revert to Step 5. If the exhaustive search procedure for the pairs is completed, proceed to Step 8.

8. If $n_c > 0$, take an element of the group ζ. Otherwise, proceed to Step 11.

9. Execute (22) through the exhaustive search procedure over the set (23).

10. Take the next element from the group ζ and revert to Step 9. If the exhaustive search procedure within the group ζ is completed, proceed to Step 11.

11. If the record value of the objective function $\hat{f}$ yielded by Steps 4–10 is less than $f^{(\ell)}_{\min} - \varepsilon$, replace the value $f^{(\ell)}_{\min}$ with $\hat{f}$ and revert to Step 4 with the corresponding new point. Otherwise, remember the objective function value $\min(f^{(\ell)}_{\min}, \hat{f})$ and the corresponding point $\tilde{x}$, take the next initial point, and revert to Step 3. If the search procedure for all initial points obtained in Step 2 is completed, proceed to Step 12.

12. Find the minimum among the objective function values obtained for all initial points and the corresponding point $\tilde{x}$. Complete the search procedure.

Additional search stages can be embedded in this algorithm if the objective function value does not decrease in Step 11: 1) increase the weight coefficients μ_1 and μ_2 and continue the search procedure from the resulting point; 2) continue the search procedure with smaller values of the set of variations (19) for α and β.

4. EXAMPLES

4.1. Underwater Vehicle Position Control

The transfer functions for local coordinate system positioning were identified in [16]. In this section, we consider control design for the coordinate z with the identified transfer function

$$P_z(s) = \frac{0.018}{s(0.98s+1)}. \tag{24}$$

The two-degree-of-freedom (2DOF) PID controller presented in [16] allows setting a desired transfer function of the closed loop system. For this example, we take the desired transfer function

$$P_m(s) = \frac{1}{(0.98s+1)(0.5s+1)}. \tag{25}$$

The denominator of the transfer function (25) must be included in the desired characteristic polynomial of the closed loop system when designing a 2DOF controller. Then there are only two roots left for variation. Assume that they form a complex conjugate pair of roots of the characteristic polynomial. In this case, the controller coefficients

$$C(s) = \frac{d_2s^2 + d_1s + d_0}{s(c_1s + c_0)} \tag{26}$$

are obtained from the equation

$$s^2(0.98s+1)(c_1s+c_0) + 0.018(d_2s^2 + d_1s + d_0) = (0.98s+1)(0.5s+1)(s^2 + 2\zeta\breve{\omega} + \breve{\omega}^2).$$

This example illustrates the search procedure for the variables ζ and $\breve{\omega}$. Since $n_a = 1$ here, we use coordinate descent instead of Algorithm 1.

Let the following bounds be specified:

$$\breve{\omega}_{\min} = 0.6,\ \breve{\omega}_{\max} = 20,\ \zeta_{\min} = 0.8,\ S_{\max} = 1.7,\ N_{\max} = 150. \tag{27}$$

Weight coefficients should be assigned for the penalty functions of the objective function (13). These coefficients are chosen so that the constraints have priority over disturbance rejection. Note that the penalty functions are included in (13) as the ratio of the H_∞ norm to its admissible

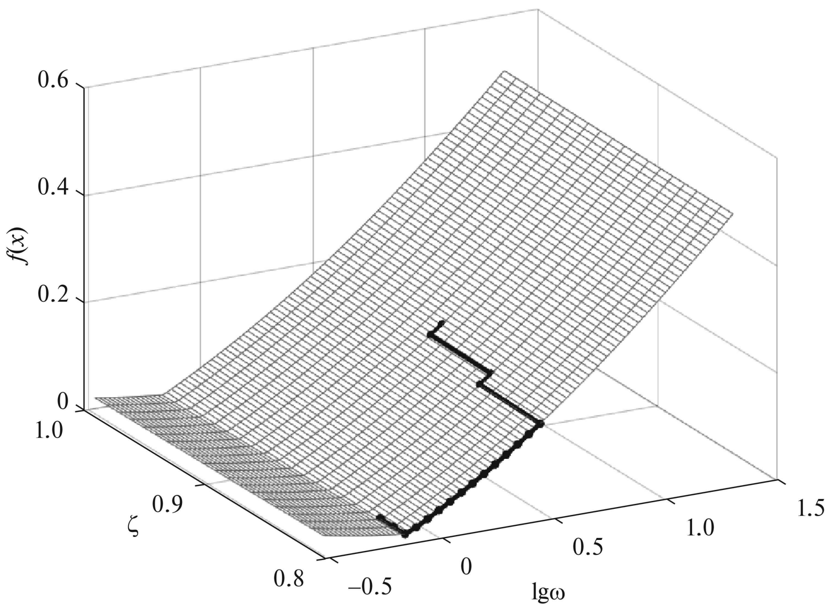

Fig. 2. Coordinate descent: $\tilde{x} = [\tilde{\omega}, \zeta]$.

maximum value whereas the H_∞ norm of the frequency response function relative to the disturbance is used in absolute units. Therefore, to choose the weight coefficients, it is necessary to estimate the value $\|G_{yf}(j\omega)\|_\infty$. For example, for the minimum values from the admissible region $\breve{\omega} = 0.6$, $\zeta = 0.8$, we obtain $\|G_{yf}(j\omega)\|_\infty = 0.0239$. Then the values $\mu_1 = 1$ and $\mu_2 = 0.1$ can be taken. The set of steps (23) is used for both variables.

Figure 2 shows the surface of the objective function on a grid with steps of 0.02 for $\tilde{\omega}$ and 0.01 for ζ within the given constraints. Also, this figure presents the objective function values in each step of the coordinate descent procedure with the initial point

$$\tilde{x}_0 = \left[\frac{\lg \breve{\omega}_{\max} + \lg \breve{\omega}_{\min}}{2}, \frac{1 + \zeta_{\min}}{2}\right] = [0.5396, 0.9].$$

The minimum point is $\breve{\omega} = 0.6928, \zeta = 0.821$, and the corresponding values are

$$\|S(j\omega)\|_\infty = 1.27, \quad \|G_{u\nu}(j\omega)\|_\infty = 149.97, \quad \|G_{yf}(j\omega)\|_\infty = 0.0206.$$

4.2. Controller for a Two-Mass System

Consider the benchmark problem presented in [17], i.e., a robust control design for two trolleys joined by a spring. For this problem, the pole placement optimization method was used to build a controller satisfying the speed and robustness requirements of the system [12]. Note that the standard global optimization procedure from the MATLAB Global Optimization Toolbox [13] was applied therein to find the optimal roots of the characteristic polynomial. In this subsection, we use Algorithm 1 to solve the same problem.

Let the transfer function relative to control be

$$P(s) = \frac{1}{s^2(s^2+2)}. \tag{28}$$

In this plant, control and disturbance are applied at different points, and the open-loop transfer function relative to the disturbance is known:

$$P_f(s) = \frac{s^2+1}{s^2(s^2+2)}. \tag{29}$$

In this case, the H_∞ norm of the closed-loop frequency transfer function relative to the disturbance differs from (7) and is calculated as

$$\|G_{yf}(j\omega)\|_\infty = \sup_\omega \left|\frac{b_f(j\omega)c(j\omega)}{\delta(j\omega)}\right|, \tag{30}$$

where $b_f(j\omega)$ is the numerator polynomial of the transfer function (29).

Similar to [12], we design a controller of the form (2) with $n = 4$ under the following bounds and constraints:

$$\lambda_{\min} = \breve{\omega}_{\min} = 0.1, \quad \lambda_{\max} = \breve{\omega}_{\max} = 100, \quad \zeta_{\min} = 0.7, \quad S_{\max} = 1.665, \quad N_{\max} = 100. \tag{31}$$

We choose the desired structure of the characteristic polynomial (5) with $n_r = 1$ and $n_c = 3$ and the weight coefficients $\mu_1 = \mu_2 = 100$ for the penalty functions in (13). Let the threshold for varying the objective function be $\varepsilon = 10^{-6}$. We form twenty-four initial points for multiple start by choosing $n_1 = 4$, $n_2 = 3$, and $n_3 = 2$ and using (16) for the groups $\tilde{\lambda}$ and $\tilde{\omega}$ as well as the following alternatives for the group ζ : 1) all elements equal $\zeta_{\min}$; 2) all elements equal one.

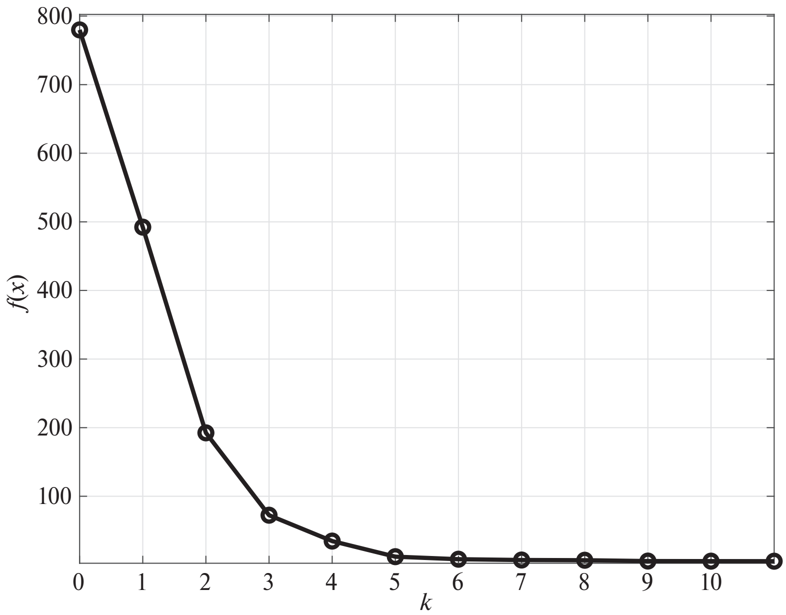

Fig. 3. The values of the objective function $f(x)$ at each iteration of the search algorithm.

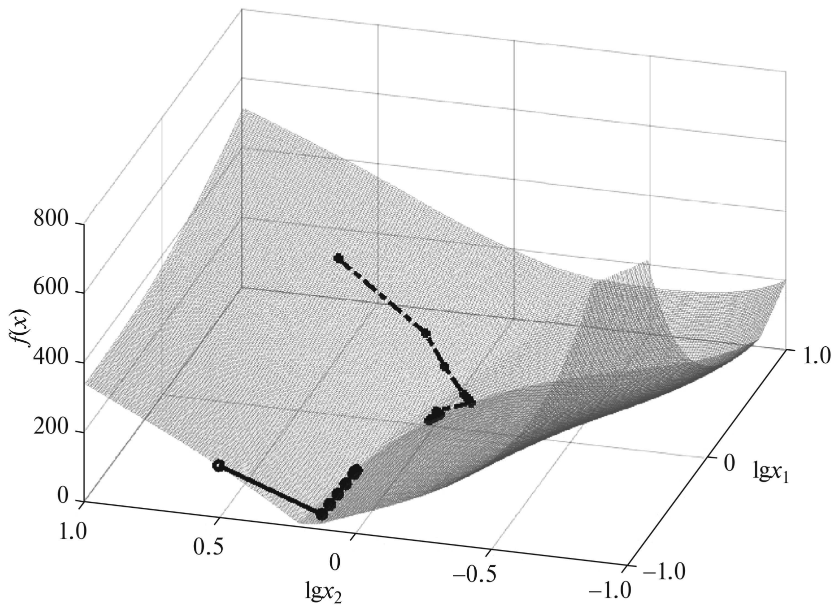

Fig. 4. The values of the objective function $f(\tilde{\lambda}_1, \tilde{\omega}_1)$.

The resulting minimum point of the objective function (13) is

$$x_{\min} = [0.3417, 1.4138, 1.4145, 3.6593, 0.701, 0.700, 0.700], \tag{32}$$

for which

$$\|S(j\omega)\|_\infty = 1.665, \quad \|G_{u\nu}(j\omega)\|_\infty = 99.96, \quad \|G_{yf}(j\omega)\|_\infty = 5.296. \tag{33}$$

The minimum was found in twenty iterations from an initial point. Figure 3 shows the graph of the record values of the objective function. Other six points of multiple start yielded $\|G_{yf}(j\omega)\|_\infty < 6$ under the valid constraints. The remaining initial points led to local minima with the invalid constraint $\|S(j\omega)\|_\infty \leqslant S_{\max}$ or a higher value of $\|G_{yf}(j\omega)\|_\infty$. Only two of the twenty-four initial points of multiple start resulted in the same local minimum; in the rest cases, the search procedure was completed at different points.

Figure 4 shows the surface of the objective function calculated for the vector $\tilde{x}$ with only the first two elements being varied on a grid (and the rest equaling the obtained values (32)) and search alternatives for these two elements from the initial points $[0.5, 0.5]$ and $[-1, 0.5]$. Obviously, the search procedure converged to different local minima with values of the objective function equal to 19.8 and 5.3, respectively. In other words, the objective function in this example has a complicated ravine surface even in the simplified case with two variables.

The same example with the same constraints was solved by several methods in [12]. The controller with $\|G_{yf}(j\omega)\|_\infty = 5.301$, the result almost coinciding with (33), was obtained using systune, the fixed-structure control system tuning procedure [18] of the MATLAB Robust Control Toolbox. The solution by the pole placement optimization method using the standard global optimization procedure was implemented in [12]; the resulting controller rejects the disturbance slightly worse, ensuring the value $\|G_{yf}(j\omega)\|_\infty = 6.64$.

Thus, the search algorithm proposed in this article found a better solution than the standard global optimization procedure. The solution obtained by systune is practically not improved, which suggests its global minimum character.

5. CONCLUSIONS

The control design problem using the pole placement method has been considered, and an algorithm has been developed to find the desired poles based on the specified system quality criteria and constraints. The value of the H_∞ norm of the frequency transfer function relative to the disturbance has been selected as the quality criterion of the system, and the maximum admissible values of the H_∞ norms of the sensitivity function and the frequency transfer function relative to the measurement noise have been set as the constraints. The resulting search algorithm can be used for other criteria and constraints. In this case, only the penalty components (12) in the objective function (13) will be changed. Note that in the example of subsection 4.1, the controller structure differs from (2) since an integral component has been added to the controller. Thus, the scope of application of the developed approach is not restricted to systems with the controller (2): it covers all controller structures that can be obtained by the pole placement method. Also, for the sake of simplicity, an exogenous disturbance has been applied along with the control action in the system structure. Indeed, the real transfer function relative to the disturbance is often unknown; in this case, such a simplification of the system structure still allows considering the effect of the disturbance in the system. If the plant's transfer function relative to the disturbance is known (see the example of subsection 4.2), it should be used when forming the transfer function of the closed loop system relative to the disturbance.

The advantages of the proposed search method are due to considering the properties of the characteristic polynomial roots. The logarithmic scale taken for the moduli of the characteristic polynomial roots provides the following benefits. First, it serves to reasonably choose the increment of the variables in the search procedure. Second, it allows one to form a limited set of initial points for the multiple start procedure. The search algorithm with a pair of simultaneously varied elements finds the minimum for an objective function with a complicated surface. Thus, the known features of the vector of variables in the problem under consideration have been utilized to develop an effective constrained minimization algorithm for a non-convex multi-extremal objective function.

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